STUDENT NAME:

Linear Algebra Graduate Comprehensive Exam, January 2016
Worcester Polytechnic Institute

Work out 4 category I problems and 1 category II problem,
or work out 3 category I problems and 2 category II problems.
Write down detailed proofs of every statement you make.
No Books. No Notes. No calculators.

Category I problems

problem I.1

Let A be the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Compute
(a) the rank of A,

(b) the trace of A,

(c) the determinant of A

(d) the characteristic polynomial of A

(e) all the eigenvalues of A

(f) find corresponding eigenvectors

(g) Compute $(2I - A)^3 - 2(2I - A)$

Does the answer to (g) surprise you? Can you formulate a relation of A to its characteristic polynomial?

(h) Is A diagonalizable? If so, find an orthogonal matrix Q such that $A = Q\Lambda Q^T$

(i) Compute $A^T A$ and AA^T and their eigenvalues and eigenvectors.

(j) Compute e^A and comment (without computing it) on the behavior of the solution to $\frac{d\vec{u}}{dt} = A\vec{u}, \vec{u}(0) = \vec{u}_0$.

problem I.2

Consider the linear map $T : \mathbb{P}_3 \to \mathbb{P}_2$ defined by differentiation, i.e., by $T(p) = p' \in \mathbb{P}_2$ for $p \in \mathbb{P}_3$. Find the matrix representation of T with respect to the bases
$\{1+x, 1-x, x+x^2, x^2-x^3\}$ for $\mathbb{P}_3$ and $\{1, x, x^2\}$ for $\mathbb{P}_2$.

problem I.3

Compute the matrix of transformation of coordinates (back and forth) from the canonical basis in $\mathbb{R}^3$ to the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(these vectors' coordinates are with respect to the canonical basis).

problem I.4

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}.$$

Find the least square solution of $Ax = b$.

problem I.5

The set of all real $n \times n$ matrices, denoted $\mathbb{R}^{n\times n}$, is a vector space under the usual operations of matrix addition and scalar multiplication. Consider $\mathcal{S} \equiv \{A \in \mathbb{R}^{n\times n} : A^T = -A\}$, the set of all skew-symmetric matrices in $\mathbb{R}^{n\times n}$.

(a) Show that $\mathcal{S}$ is a subspace of $\mathbb{R}^{n\times n}$.

(b) Show that $P : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by $P(A) = \frac{1}{2}\left(A - A^T\right)$ is the projection of $\mathbb{R}^{n\times n}$ onto $\mathcal{S}$ that is orthogonal with respect to the Frobenius inner product.

Category II problems

problem II.1

Prove that if A is a non-singular $n \times n$ matrix, then there exists a polynomial $f(t)$ such that $Af(A) = f(A)A = I$.

problem II.2

Prove that any square $n \times n$ matrix A can be obtained as a limit of matrices $A_l \to A$ that have n distinct eigenvalues.